Stochastic dynamics from the fractional Fokker-Planck-Kolmogorov equation: Large-scale behavior of the turbulent transport coefficient

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The formulation of the fractional Fokker-Planck-Kolmogorov (FPK) equation [Physica D **76**, 110 (1994)] has led to important advances in the description of the stochastic dynamics of Hamiltonian systems. Here, the long-time behavior of the basic transport processes obeying the fractional FPK equation is analyzed. A derivation of the large-scale turbulent transport coefficient for a Hamiltonian system with $1\frac{1}{2}$ degrees of freedom is proposed in connection with the fractal structure of the particle chaotic trajectories. The principal transport regimes (i.e., a diffusion-type process, ballistic motion, subdiffusion in the limit of the frozen Hamiltonian, and behavior associated with self-organized criticality) are obtained as partial cases of the generalized transport law. A comparison with recent numerical and experimental studies is given.

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From a wealth of studies it has been clearly established [1,2] that the Hamiltonian chaotic dynamics of passive particles can be adequately described by the fractional extension of the Fokker-Planck-Kolmogorov (FPK) equation

$$\frac{\partial^\beta P}{\partial t^\beta}=\frac{\partial^\alpha}{\partial(-\xi)^\alpha}(\mathcal{A}P)+\frac{1}{2}\,\frac{\partial^{2\alpha}}{\partial(-\xi)^{2\alpha}}(\mathcal{B}P), \tag{1}$$

where $P\equiv P(\xi,t)$ is the probability density of finding a particle at point ξ at time t, and the fractional exponents (α,β) define the derivatives over space (ξ) and time (t) variables, respectively. The quantities $\mathcal{A}$ and $\mathcal{B}$ on the right of Eq. (1) are given by

$$\mathcal{A}=\lim_{\Delta t\to 0}\frac{\langle\langle|\Delta\xi|^\alpha\rangle\rangle}{(\Delta t)^\beta},\quad \mathcal{B}=\lim_{\Delta t\to 0}\frac{\langle\langle|\Delta\xi|^{2\alpha}\rangle\rangle}{(\Delta t)^\beta}, \tag{2}$$

where $\langle\langle\cdots\rangle\rangle$ denotes a generalized convolution operator [2]. Equations (1) and (2) reproduce the standard FPK equation [2,3] for $\alpha=\beta=1$.

The introduction of the fractional parameters (α,β) in the kinetic equation (1) accounts for the rich class of anomalous dynamical phenomena such as long-time trappings and almost regular bursts like Levy flights [2,4]. A comprehension of the essential role played by trappings and bursts in the description of chaos led to the formulation of the ''strange kinetics'' discussed in Ref. [5].

The fractional FPK equation (1) admits scale-invariant (i.e., self-similar) solutions in either of the partial cases $\{\mathcal{A}=0,\mathcal{B}\neq 0\}$ and $\{\mathcal{A}\neq 0,\mathcal{B}=0\}$. In our study, we are mostly interested in the case $\{\mathcal{A}=0,\mathcal{B}\neq 0\}$ when Eq. (1) is reduced to the fractional transport equation

$$(\partial^\beta P)/(\partial t^\beta)=\tfrac{1}{2}\,\{\partial^{2\alpha}/[\partial(-\xi)^{2\alpha}]\}(\mathcal{B}P). \tag{3}$$

Equation (3) includes both Levy flights [4] and diffusion phenomena on fractal sets [6,7]. The asymptotics of the transport processes deriving from Eq. (3) are given by [2]

$$\langle|\xi|^2\rangle=2\mathcal{D}t^{\beta/\alpha}\quad(t\to\infty), \tag{4}$$

where $\mathcal{D}$ is the generalized transport coefficient. The explicit form of $\mathcal{D}$ depends on the value of the anomalous scaling exponent β/α assumed in Eq. (4). The effect of the *topology* of phase space of a chaotic system on the basic transport law (4) was addressed in Refs. [2,6]. Setting $\alpha=\beta=1$ in expression (4), one recovers the diffusion-type process

$$\langle|\xi|^2\rangle=2\mathcal{D}t^1\quad(t\to\infty) \tag{5}$$

associated with the conventional FPK equation [3].

The anomalous transport law (4) is customarily represented in the form [8]

$$\langle|\xi|^2\rangle=2\mathcal{D}t^{2H}\quad(t\to\infty), \tag{6}$$

where $0\leqslant H\leqslant 1$ is the Hurst exponent. A comparison with Eq. (4) shows that $H=\beta/2\alpha$. As was demonstrated by Mandelbrot [9], the Hurst exponent $0\leqslant H\leqslant 1$ defines the Hausdorff fractal dimension d_w of the particle chaotic trajectories, i.e.,

$$d_w=1/H,\quad d_w\geqslant 1. \tag{7}$$

In the case of diffusion (5), $\alpha=\beta=1$; hence $H=1/2$ and $d_w=2$. Combining Eqs. (6) and (7), one gets [6,10]

$$\langle|\xi|^2\rangle=2\mathcal{D}t^{2/d_w}\quad(t\to\infty). \tag{8}$$

The consideration below is restricted to a stochastic Hamiltonian system with $1\frac{1}{2}$ degrees of freedom, i.e.,

$$\frac{dx}{dt}=\frac{\partial\Phi(x,y,t)}{\partial y},\quad \frac{dy}{dt}=-\frac{\partial\Phi(x,y,t)}{\partial x}, \tag{9}$$

where $\Phi(x,y,t)$ is the time-dependent Hamiltonian function, and $\{\xi\equiv(x,y)\}$ is the phase space. Without loss of generality, we assume that $\Phi(x,y,t)$ is a periodic function of t, with the characteristic frequency ω. Our interest in Eqs. (9) is motivated by their importance for the foundations of stochastic dynamics [1–3], as well as their abundance in various applications [11].

The physical origin of stochasticity in system (9) is the *nonlinear resonance* between the periodic variation of the Hamiltonian function $\Phi(x,y,t)$ and the particle migration along the closed isoenergetic orbits $\{\Phi(x,y,t)=h\}$ [3]. (Here, the parameter h defines the corresponding energy level.) In the *low-frequency limit* $(\omega\to 0)$, the resonance conditions are satisfied near the percolating orbits [i.e., the *separatrices* $\{\Phi(x,y,t)=h_c\}$] whose diverging lengths match the particle excursion periods of $\sim 2\pi/\omega\to\infty$. As is well known [3], the resonances strongly overlap near the separatrix, and the particle motion evolves into chaotic wandering from one resonance to another. The separatrix is thus always surrounded by a layer of stochastic dynamics.

Let $\mathcal{N}\gg 1$ be the effective number of mutually overlapping resonances. The value of $\mathcal{N}$ can be estimated from first principles [i.e., from the Hamiltonian $\Phi(x,y,t)$] [3]. By order of magnitude,

$$\mathcal{N}\sim\lambda/\delta\lambda\gg 1, \tag{10}$$

where λ is the coherence scale of the field $\Phi(x,y,t)$ and $\delta\lambda$ is the characteristic distance between the resonances.

Then let us introduce the dimensionless parameter

$$A\equiv u/\omega\lambda\,, \tag{11}$$

where $u\sim[(\partial\Phi/\partial x)^2+(\partial\Phi/\partial y)^2]^{1/2}$ is the particle migration velocity. The quantity A is customarily referred to as the ''Kubo number.'' In view of the condition $\omega\to 0$ (to be implied in what follows), we have $A\to\infty$. From Eqs. (10) and (11) one gets

$$u/(\omega\delta\lambda)=A\mathcal{N}\gg A\to\infty. \tag{12}$$

Conditions (10) and (12) guarantee the transition to chaotic dynamics near the separatrix at time scales much shorter than $\sim 1/\omega$. In fact, the characteristic decorrelation time for a particle random walk on the set of resonances can be estimated as

$$\tau_c\sim C_0\frac{\delta\lambda}{u}\sim C_0\frac{\lambda}{\mathcal{N}u}\sim\frac{C_0}{A\mathcal{N}}\frac{1}{\omega}\ll\frac{1}{\omega}, \tag{13}$$

where $C_0\sim 3$ is a customarily assumed constant. The transition to chaos near the separatrix thus occurs much faster than the Hamiltonian $\Phi(x,y,t)$ evolves in time.

We now demonstrate that the fractal structure of the particle chaotic trajectories can appear in the anomalous scaling of the transport coefficient $\mathcal{D}$ [see the right hand side of Eq. (8)] versus the Kubo number A.

Consider a continuous set of initial conditions $\{(x_i^{[0]},y_i^{[0]})\}$, $i\in I$, for the Hamiltonian trajectories $\{(x_i^{[t]},y_i^{[t]})\}$, $t\geq 0$. (The subscript i denotes a continuous parameter ranging through an interval I.) The set $\{(x_i^{[0]},y_i^{[0]})\}$ can be associated with the correlation scale

$$\varepsilon\sim\max_{i,j\in I}[(x_i^{[0]}-x_j^{[0]})^2+(y_i^{[0]}-y_j^{[0]})^2]^{1/2}. \tag{14}$$

We assume that $\delta\lambda\lesssim\varepsilon\ll\lambda$: This defines a set of initially close trajectories whose further mixing and divergence support the turbulent particle transport on large scales.

For arbitrary $\delta\lambda\lesssim\varepsilon\ll\lambda$, one finds the mixing scale l where the particle trajectories become braided due to the wandering over the set of the overlapping resonances. At scales exceeding $\sim C_0\delta\lambda$, the trajectories start to reveal the property of self-similarity [2,3,12]; this enables one to relate the mixing scale l to the parameter ε via a dimensionless (i.e., power-law) function, $l\sim\varepsilon^{-\mu}$, where μ is a constant of the order of 1. More precisely,

$$l/\delta\lambda=(\varepsilon/\delta\lambda)^{-\mu}. \tag{15}$$

(We chose $\delta\lambda$ to be a suitable measure of length.) Given the mixing scale l, one estimates the corresponding fractal length L_l of the self-similar trajectories (each having the Hausdorff dimension $d_w\geq 1$) as [8,9]

$$L_l=\delta\lambda[(l/\delta\lambda)^{d_w}]. \tag{16}$$

Next, consider the entire ensemble, $\{(x_i^{[t]},y_i^{[t]})\}$, $i\in I$, of the trajectories (16) at the mixing scale l:

$$\{[(x_i^{[t]}-x_i^{[0]})^2+(y_i^{[t]}-y_i^{[0]})^2]^{1/2}\sim l\}. \tag{17}$$

These cover a phase volume

$$\mathcal{V}_l=\varepsilon L_l \tag{18}$$

equal to the product of the correlation scale ε with the trajectory length L_l. Substituting ε for $\delta\lambda(l/\delta\lambda)^{-1/\mu}$ [see Eq. (15)], we get

$$\mathcal{V}_l=\delta\lambda^2[(l/\delta\lambda)^{d_w-1/\mu}]. \tag{19}$$

On the other hand, the self-similarity condition applied to the entire ensemble (17) reads $L_l/\delta\lambda\sim l/\varepsilon$, yielding $\mathcal{V}_l\sim\varepsilon L_l\sim l\delta\lambda$. This is consistent with Eq. (19) if the parameter μ obeys

$$1/\mu=d_w-1\geq 0. \tag{20}$$

In view of the explicit time dependence of the Hamiltonian $\Phi(x,y,t)$, the ensemble of chaotic trajectories (17) observes the finite lifetime

$$\tau\sim\varepsilon/\omega\lambda\,, \tag{21}$$

where ω describes the variation of the field $\Phi(x,y,t)$ at the (relatively large) coherence scales $\sim\lambda\gg\delta\lambda$.

We now require that the particles migrating with the characteristic velocity u cover a distance of the order of L_l during the lifetime of the ensemble τ:

$$L_l\sim u\tau. \tag{22}$$

Combining Eqs. (15),(16) and (20),(22), after simple algebra one obtains

$$A=u/\omega\lambda=(l/\delta\lambda)^{2d_w-1} \tag{23}$$

where A is the Kubo number. Using Eqs. (13), (15), (20), (21), and (23), it is straightforward to verify that the lifetime τ far exceeds the microscopic decorrelation time τ_c for $A \to\infty$, i.e.,

$$\tau \gg \tau_c . \tag{24}$$

Condition (24) shows that the stochastic regime for the set of the initial conditions (14) is achieved long before the ensemble $\{(x_i^{[t]}, y_i^{[t]})\}$ decays due to the substantial nonautonomy of the Hamiltonian $\Phi(x,y,t)$ on long time scales $\sim 1/\omega$. (Strictly speaking, the stochasticity is *intermittent* in time since the chaotic trajectories can sporadically leave the close-to-separatrix domain where the resonances overlap. The effect of intermittency on the particle chaotic dynamics was analyzed in Ref. [13] in connection with the role of coherent vertical structures in the flow.)

The large-scale turbulent transport coefficient for the ensemble of chaotic trajectories (17) can be determined as

$$\mathcal{D} \sim (1/2\tau)\mathcal{V}_l \mathcal{N} \omega^{-1+2/d_w}, \tag{25}$$

where $\mathcal{N} \gg 1$ is the effective number of mutually overlapping resonances, and the factor ω^{-1+2/d_w} stands for the anomalous behavior of $\langle|\xi|^2\rangle \propto t^{2/d_w}$ versus time t [see Eqs. (2) and (8)]. Considering Eqs. (10), (16), (18), (21), and (23), we find

$$\mathcal{D} \sim \tfrac{1}{2} \lambda^2 \omega^{2/d_w} A^{d_w/(2d_w-1)} \tag{26}$$

independently of the microscopic scale $\delta\lambda$.

The proposed derivation of the turbulent transport coefficient (26) generalizes the pioneering approach of Gruzinov *et al.* [14], who considered anomalous particle diffusion on weblike convective structures (i.e., the ''*a* webs'' [11]). The Hausdorff fractal dimension of the *a* webs was conjectured to coincide with the hull exponent $d_h = 7/4$ describing the external perimeters of the percolating isoenergetic contours [11,14]. Setting $d_w = d_h = 7/4$ in expression (26), one reproduces the scaling of the turbulent diffusion coefficient, $\mathcal{D} \propto A^{7/10}$, obtained in Ref. [14]. In our study, we avoid direct associations between the dynamical chaos in Eq. (9) and the structural properties of the isoenergetic contours, in contrast to [14]. This modification addresses the substantially dynamical nature of the fractal dimension d_w.

In view of expression (26), the anomalous transport law (8) becomes

$$\langle|\xi(t)|^2\rangle \sim \lambda^2 \omega^{2H} A^{\gamma} t^{2H} \quad (t\to\infty), \tag{27}$$

where $A\to\infty$; $H = 1/d_w$ is the Hurst exponent; and

$$\gamma = d_w/(2d_w - 1) = 1/(2-H) \tag{28}$$

determines the scaling of the transport coefficient $\mathcal{D}$ with the Kubo number A.

For vanishing frequency $\omega\to 0$, the transport coefficient (26) goes to zero, $\mathcal{D}\to 0$, if the fractal dimension d_w satisfies the condition $2/d_w > d_w/(2d_w-1)$, i.e.,

$$1 \leqslant d_w < 2+\sqrt{2}. \tag{29}$$

Hence, the Hurst exponent H obeys

$$(2-\sqrt{2})/2 < H \leqslant 1, \tag{30}$$

while the Kubo number exponent γ ranges through

$$2-\sqrt{2} < \gamma \leqslant 1. \tag{31}$$

In the limiting case $d_w \to 2+\sqrt{2}$, $H\to(2-\sqrt{2})/2$, $\gamma \to 2-\sqrt{2}$, the transport coefficient $\mathcal{D}$ saturates at

$$\mathcal{D} = \tfrac{1}{2} \lambda^2 \omega^{2-\sqrt{2}} A^{2-\sqrt{2}} = \tfrac{1}{2} \lambda^{\sqrt{2}} u^{2-\sqrt{2}} \tag{32}$$

and does not depend on ω. The corresponding transport law is the *subdiffusion*

$$\langle|\xi(t)|^2\rangle \sim 2\mathcal{D} t^{2-\sqrt{2}} \quad (t\to\infty) \tag{33}$$

with exponent $2H = 2-\sqrt{2} \sim 0.6$. This regime can be identified with particle dissemination in the static (i.e., frozen) field $\Phi(x,y,\omega=0)$ due to the intrinsically unstable dynamics near the separatrix [3].

One remarks that the value of $2H$ in Eq. (27) satisfies

$$2H \geqslant \gamma. \tag{34}$$

The exact equality in Eq. (34) is achieved for the frozen Hamiltonian $\Phi(x,y,\omega=0)$. Condition (34) guarantees the finiteness of the turbulent diffusion coefficient $\mathcal{D}$ for vanishing ω.

The opposite limit, $d_w = 1$, $H = 1$, $\gamma = 1$, determined by conditions (29)–(31), describes the *ballistic* type of particle dynamics:

$$\mathcal{D} = \tfrac{1}{2} \lambda^2 \omega^2 A^1 = \tfrac{1}{2} \omega\lambda u, \tag{35}$$

$$\langle|\xi(t)|^2\rangle \sim 2\mathcal{D} t^2 \quad (t\to\infty). \tag{36}$$

The ballistic regime (35), (36) can be associated with the dominant role of *Levy flights* and was analyzed numerically in, e.g., Ref. [15].

The *diffusion* process (5) can be recovered from relations (26) and (27) by setting $H = 1/2$ and $d_w = 2$. Hence, $\gamma = 2/3$,

$$\mathcal{D} \sim \tfrac{1}{2} \omega\lambda^2 A^{2/3}, \tag{37}$$

$$\langle|\xi(t)|^2\rangle \sim \tfrac{1}{2} \omega\lambda^2 A^{2/3} t^1 \quad (t\to\infty). \tag{38}$$

The diffusion-type transport (37),(38) can be realized for relatively strong stochasticity when the typical scales of the chaotic domains are comparable with $\lambda \gg \delta\lambda$ or more [2]. Note that the Kubo number exponent $\gamma = 2/3$ is actually close to (although slightly smaller than) the original estimate $\gamma = 7/10$ proposed by Gruzinov *et al*. [14]. The deviation of γ from the value 7/10 was recognized numerically by Reuss *et al.* [16] and Zimbardo *et al.* [17], who observed a somewhat slower dependence of the turbulent transport coefficient upon the Kubo number as compared with the prediction of Gruzinov *et al*. [14], $\mathcal{D} \propto A^{7/10}$.

In conclusion, let us consider the anomalous transport law (26),(27) in association with the particle dynamics near self-organized criticality (SOC). The concept of SOC [18] addresses the important issue of self-organization in nonlinear dynamical systems in connection with the near-critical behavior characteristic of many phenomena [19]. A transition to SOC customarily leads to the formation of self-similar (fractal) structures in a wide range of scales. Real-space properties of these structures can often be described by the generalized (Hausdorff) dimension d_f, which deviates from the topological (integer) dimension of the system. In the frequency domain, the fractal geometry of SOC systems appears in the power-law decrease $f^{-\eta}$ of the Fourier energy density spectrum [18], where the power exponent $\eta = 2d_f - 1$ depends on the Hausdorff dimension d_f [20]. Self-organized criticality thus establishes an intimate connection between spatial scales and time scales for complex dynamical systems. (Note that the topological characteristics of SOC systems must obey the condition of path connectedness. In general, the features of connectedness of the fractal distributions are not in one-to-one correspondence with the shape of the Fourier energy density spectrum as determined by η [21].) In the context of transport processes, the notion of SOC can be considered as synonymous with ''self-organization to a state of critical percolation.'' For critical percolation on a plane, $1 \leqslant d_f \leqslant \mathcal{S}$ [22] where $\mathcal{S} = \log_{10} 8/\log_{10} 3 \approx 1.89$ is the Hausdorff dimension of the square Sierpinski carpet [8]. Hence $1 \leqslant \eta \leqslant 2\mathcal{S} - 1 \approx 2.78$. The limiting value $d_f = 1$ corresponds to minimally developed fractal structures in space and the f^{-1} spectrum often referred to as Flicker noise. The critical percolation regime implies a ''universal'' relationship between the Hausdorff dimension of the chaotic trajectories d_w and the fractal dimension d_f of the structure on which the random walk takes place:

$$2d_f/d_w = \mathcal{C} \approx 1.327 \sim 4/3. \tag{39}$$

(The ratio $d_s \equiv 2d_f/d_w$ is often termed the *spectral* fractal dimension [6,10].) Equation (39) represents an improved form of the Alexander-Orbach conjecture [23] and was substantiated in Ref. [22]. The quantity $\mathcal{C}$ is the percolation constant [21], the fundamental topological parameter deriving from the transcendental algebraic equation $\mathcal{C}\pi^{\mathcal{C}/2}\Gamma(\mathcal{C}/2+1) = \pi$ [22]. (Here, Γ is the Euler gamma function.) Replacing d_w by $2d_f/\mathcal{C}$ in Eqs. (26) and (27), one arrives at the generalized transport law for SOC:

$$\mathcal{D} \sim \tfrac{1}{2}\lambda^2 \omega^{\mathcal{C}/d_f} A^{2d_f/(4d_f - \mathcal{C})}, \tag{40}$$

$$\langle |\xi(t)|^2 \rangle \sim 2\mathcal{D} t^{\mathcal{C}/d_f} \quad (t \to \infty). \tag{41}$$

The particle transport at SOC is thus superdiffusive for $1 \leqslant d_f < \mathcal{C}$ and subdiffusive for $\mathcal{C} < d_f \leqslant \mathcal{S}$. The value $d_f = \mathcal{C}$ reproduces the diffusion-type behavior (37),(38) and corresponds to the ''Kolmogorov'' spectrum $f^{-\eta}$ where $\eta = 2\mathcal{C} - 1 \approx 1.65 \sim 5/3$. The upper limit $d_f = \mathcal{S}$ supports the subdiffusive regime with the spectral index $\eta = 2\mathcal{S} - 1 \approx 2.78$, the Hurst exponent $H = \mathcal{C}/2\mathcal{S} \approx 0.35$, and the Kubo number exponent $\gamma = 2\mathcal{S}/(4\mathcal{S} - \mathcal{C}) \approx 0.61$. Finally, the value $d_f = 1$ (related to the Flicker noise f^{-1}) leads to the Hurst exponent $H = \mathcal{C}/2 \sim 2/3$ and the Kubo number exponent $\gamma = 2/(4 - \mathcal{C}) \sim 3/4$. The estimate $H \sim 2/3$ is in accord with the results of Carreras *et al.* [24,25] who found $H \sim 0.6$–0.75 for the SOC-associated currents in tokamaks. Moreover, Carreras *et al.* [25] observed a fluctuation spectrum close to f^{-1} just in the self-similarity region where superdiffusive transport with $H \sim 0.6$–0.75 was recognized. These intriguing results might help uncover the fundamental relationship between the statistical properties of SOC systems and the basic transport mechanisms operating therein.

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